# Extremal black holes in $D = 5$: SUSY vs. Gauss-Bonnet corrections

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Abstract: We analyse near-horizon solutions and compare the results for the black hole entropy of five-dimensional spherically symmetric extremal black holes when the $N = 2$ SUGRA actions are supplied with two different types of higher-order corrections: (1) supersymmetric completion of gravitational Chern-Simons term, and (2) Gauss-Bonnet term. We show that for large BPS black holes lowest order $\alpha'$ corrections to the entropy are the same, but for non-BPS are generally different. We pay special attention to the class of prepotentials connected with $K3 \times T^2$ and $T^6$ compactifications. For supersymmetric correction we find beside BPS also a set of non-BPS solutions. In the particular case of $T^6$ compactification (equivalent to the heterotic string on $T^4 \times S^1$) we find the (almost) complete set of solutions (with exception of some non-BPS small black holes), and show that entropy of small black holes is different from statistical entropy obtained by counting of microstates of heterotic string theory. We also find complete set of solutions for $K3 \times T^2$ and $T^6$ case when correction is given by Gauss-Bonnet term. Contrary to four-dimensional case, obtained entropy is different from the one with supersymmetric correction. We show that in Gauss-Bonnet case entropy of small "BPS" black holes agrees with microscopic entropy in the known cases.

Keywords: Black Holes in String Theory, Black Holes, Supergravity Models.

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Contents

1. Introduction

In recent years a lot of attention was directed towards higher curvature corrections in effective SUGRA field theories appearing in compactifications of string theories. Particularly interesting question is how these corrections are affecting black hole solutions, and in particular their entropies. One of the main successes so far of string theory is that it offers statistical explanation of black hole entropy by direct counting of microstates. In some cases it was possible to obtain not only lowest order Bekenstein-Hawking area law, but also higher corrections in string tension $\alpha'$, and even $\alpha'$ exact expressions for the entropy. These calculations are typically performed in the limit of small string coupling constant $g_s$ in the realm of perturbative string theory, where space-time is almost flat and black holes are actually not present. It is expected that these objects become black holes when one turns on $g_s$ enough so that their size becomes smaller than their corresponding Schwarzschild radius. Unfortunately, it is not known how to make direct calculations in string theory in this regime. However, when one goes in the opposite extreme where the Schwarzschild

radius becomes much larger than the string length $\ell_s = \sqrt{\alpha'}$, then one can use low energy effective action where black holes appear as classical solutions.

The situation is especially interesting for BPS black holes. In this case on the perturbative string side one is counting number of states in short multiplets, which is expected to not depend on $g_s$, at least generically (this property can be violated in special circumstances like, e.g., short multiplet crossings). This means that one can directly compare statistical (or microscopic) entropy from perturbative string and macroscopic entropy from classical supergravity. By comparing the results from the both limits we have not only succeeded to do sophisticated perturbative consistency checks on the theory, but also improved our understanding both of string theory and supergravity. Developments include attractor mechanism and relation to topological strings [1]. Especially fruitful and rich are results obtained for black holes in $D = 4$ (for reviews see [2]).

In $D = 4$ especially nice examples are provided by heterotic string compactified on $K3 \times S^1 \times S^1$ or $T^4 \times S^1 \times S^1$ [3]. The simplest BPS states correspond to large spherically symmetric black holes having 4 charges (2 electric and 2 magnetic), for which statistical entropy was found [4–7]. The macroscopic black hole entropy was calculated using two types of actions with higher order $R^2$ terms — supersymmetric and Gauss-Bonnet. In the regime where $g_s$ is small near the horizon (limit where electric charges are much larger than magnetic) all results are *exactly* equal (i.e., in all orders in $\alpha'$). This is surprising because in both of these effective actions one has neglected an infinite number of terms in low energy effective action and one would at best expect agreement in first order in $\alpha'$. There is an argumentation [8, 9], based on $AdS_3$ arguments, which explains why corrections of higher order than $R^2$ are irrelevant for calculation of black hole entropy, but it still does not explain why these two particular types of corrections are working for BPS black holes.

These matches are even more surprising when one takes magnetic charges to be zero. One gets 2-charge *small* black holes which in the lowest order have null-singular horizon with vanishing area, which is made regular by inclusion of higher curvature corrections [10, 11]. As curvature is of order $1/\alpha'$, all terms in the effective action give a priori contribution to the entropy which is of the same order in $\alpha'$. This is a consequence of the fact that here we are naively outside of the regime where effective action should be applicable.

In view of these results, it would be interesting to consider what happens in higher dimensions $D > 4$. 2-charge BPS states and corresponding small extremal black holes generalize to all $D \leq 9$. In [12] it was shown that simple Gauss-Bonnet correction gives correct result for the entropy of such black holes also in $D = 5$, but not for $D > 5$. Afterwards, in [13] it was shown that there is an effective action where higher order terms are given by linear combination of all generalized Gauss-Bonnet densities (with uniquely fixed coefficients) which gives the correct entropy for all dimensions. For large black holes things do not generalize directly. In $D = 5$ simplest are 3-charge BPS black holes, but even for them statistical entropy is known only in lowest order in $\alpha'$ [14]. Let us mention that the argumentation based on $AdS_3$ geometry has not been generalized to $D > 4$.

Motivated by all this, in this paper we analyse near-horizon solutions and calculate macroscopic entropy for a class of five-dimensional black holes in the $N = 2$ supergravities for which higher-derivative $R^2$ actions were recently obtained in [15]. In section 2 we

present $D = 5$ supersymmetric action [15]. In section 3 we review Sen's entropy function formalism [16]. In section 4 we present maximally supersymmetric $AdS_2 \times S^3$ solution which describes near-horizon geometry of purely electrically charged 1/2 BPS black holes. In section 5 for the case of simple $STU$ prepotential we find non-BPS solutions for all values of charges, except for some small black holes with one charge equal to 0 or $\pm 1$. In section 6 we show how and when solutions from section 5 can be generalized. In section 7 we present near horizon solutions for 3-charge black holes in heterotic string theory compactified on $K3 \times S^1$ when the $R^2$ correction is given by Gauss-Bonnet density. and compare them with the results from SUSY action. We show that for small black holes Gauss-Bonnet correction keeps producing results in agreement with microscopic analyses. In appendix A we present generalisation of section 5 to general correction coefficients $c_I$, and in appendix B derivations of results presented in section 7.

While our work was in the late stages references [17, 18] appeared which have some overlap with our paper. In these papers near-horizon solutions and the entropy for BPS black holes for supersymmetric corrections were given, which is a subject of our section 4. Our results are in agreement with those in [17, 18]. However, we emphasize that our near-horizon solutions in sections 5 and 6 for non-BPS black holes are completely new. Also, in [18] there is a statement on matching of the entropy of BPS black hole for supersymmetric and Gauss-Bonnet correction. We explicitly show in section 7 that this is valid just for first $\alpha'$ correction.

## 2. Higher derivative $N = 2$ SUGRA in $D = 5$

Bosonic part of the Lagrangian for the $N = 2$ supergravity action in five dimensions is given by

$$4\pi^2 \mathcal{L}_0 = 2\partial^a \mathcal{A}_i^\alpha \partial_a \mathcal{A}_\alpha^i + \mathcal{A}^2 \left( \frac{D}{4} - \frac{3}{8}R - \frac{v^2}{2} \right) + \mathcal{N} \left( \frac{D}{2} + \frac{R}{4} + 3v^2 \right) + 2\mathcal{N}_I v^{ab} F_{ab}^I$$
$$+ \mathcal{N}_{IJ} \left( \frac{1}{4} F_{ab}^I F^{J\,ab} + \frac{1}{2} \partial_a M^I \partial^a M^J \right) + \frac{e^{-1}}{24} c_{IJK} A_a^I F_{bc}^J F_{de}^K \epsilon^{abcde} \tag{2.1}$$

where $\mathcal{A}^2 = \mathcal{A}_{i\,ab}^\alpha \mathcal{A}_\alpha^{i\,ab}$ and $v^2 = v_{ab} v^{ab}$. Also,

$$\mathcal{N} = \frac{1}{6} c_{IJK} M^I M^J M^K, \quad \mathcal{N}_I = \partial_I \mathcal{N} = \frac{1}{2} c_{IJK} M^J M^K, \quad \mathcal{N}_{IJ} = \partial_I \partial_J \mathcal{N} = c_{IJK} M^K \tag{2.2}$$

A bosonic field content of the theory is the following. We have Weyl multiplet which contains the fünfbein $e_\mu^a$, the two-form auxiliary field $v_{ab}$, and the scalar auxiliary field $D$. There are $n_V$ vector multiplets enumerated by $I = 1, \ldots, n_V$, each containing the one-form gauge field $A^I$ (with the two-form field strength $F^I = dA^I$) and the scalar $M^I$. Scalar fields $\mathcal{A}_\alpha^i$, which are belonging to the hypermultiplet, can be gauge fixed and the convenient choice is given by

$$\mathcal{A}^2 = -2, \qquad \partial_a \mathcal{A}_i^\alpha = 0 \tag{2.3}$$

One can use equations of motion for auxiliary fields to get rid of them completely and obtain the Lagrangian in a standard form:

$$4\pi^2 \mathcal{L}_0 = R - G_{IJ}\partial_a M^I \partial^a M^J - \frac{1}{2} G_{IJ} F_{ab}^I F^{\mathrm{J}ab} + \frac{e^{-1}}{24} c_{IJK} A_a^I F_{bc}^J F_{de}^K \epsilon^{abcde} \qquad (2.4)$$

with

$$G_{IJ} = -\frac{1}{2}\partial_I \partial_J (\ln \mathcal{N}) = \frac{1}{2}(\mathcal{N}_I \mathcal{N}_J - \mathcal{N}_{IJ}) \qquad (2.5)$$

and where $\mathcal{N} = 1$ is implicitly understood (but only after taking derivatives in (2.5)). We shall later use this form of Lagrangian to make connection with heterotic string effective actions.

Lagrangian (2.4) can be obtained from 11-dimensional SUGRA by compactifying on six-dimensional Calabi-Yau spaces $(CY_3)$. Then $M^I$ have interpretation as moduli (volumes of $(1,1)$-cycles), and $c_{IJK}$ as intersection numbers. Condition $\mathcal{N} = 1$ is a condition of real special geometry. For a recent review and further references see [19].

Action (2.1) is invariant under SUSY variations, which when acting on the purely bosonic configurations (and after using (2.3)) are given with

$$\delta\psi_\mu^i = \mathcal{D}_\mu \varepsilon^i + \frac{1}{2} v^{ab}\gamma_{\mu ab}\varepsilon^i - \gamma_\mu \eta^i$$
$$\delta\xi^i = D\varepsilon^i - 2\gamma^c\gamma^{ab}\varepsilon^i \mathcal{D}_a v_{bc} - 2\gamma^a \varepsilon^i \epsilon_{abcde} v^{bc} v^{de} + 4\gamma \cdot v\eta^i$$
$$\delta\Omega^{Ii} = -\frac{1}{4}\gamma \cdot F^I \varepsilon^i - \frac{1}{2}\gamma^a \partial_a M^I \varepsilon^i - M^I \eta^i$$
$$\delta\zeta^\alpha = \left(3\eta^j - \gamma \cdot v\varepsilon^j\right) \mathcal{A}_j^\alpha \qquad (2.6)$$

where $\psi_\mu^i$ is gravitino, $\xi^i$ auxiliary Majorana spinor (Weyl multiplet), $\delta\Omega^{Ii}$ gaugino (vector multiplets), and $\zeta^\alpha$ is a fermion field from hypermultiplet.

In [15] four derivative part of the action was constructed by supersymmetric completion of the mixed gauge-gravitational Chern-Simons term $A \wedge \mathrm{tr}(R \wedge R)$. The bosonic part of the action relevant for our purposes was shown to be

$$4\pi^2 \mathcal{L}_1 = \frac{c_I}{24} \left\{ \frac{e^{-1}}{16}\epsilon_{abcde} A^{Ia} C^{bcfg} C^{de}{}_{fg} + M^I \left[ \frac{1}{8} C^{abcd} C_{abcd} + \frac{1}{12} D^2 - \frac{1}{3} C_{abcd} v^{ab} v^{cd} \right. \right.$$
$$+ 4v_{ab}v^{bc}v_{cd}v^{da} - (v_{ab}v^{ab})^2 + \frac{8}{3}v_{ab}\hat{\mathcal{D}}^b \hat{\mathcal{D}}_c v^{ac} + \frac{4}{3}\hat{\mathcal{D}}^a v^{bc} \hat{\mathcal{D}}_a v_{bc} + \frac{4}{3}\hat{\mathcal{D}}^a v^{bc} \hat{\mathcal{D}}_b v_{ca}$$
$$\left. -\frac{2}{3}e^{-1}\epsilon_{abcde}v^{ab}v^{cd}\hat{\mathcal{D}}_f v^{ef} \right] + F^{Iab} \left[ \frac{1}{6}v_{ab}D - \frac{1}{2}C_{abcd}v^{cd} + \frac{2}{3}e^{-1}\epsilon_{abcde}v^{cd}\hat{\mathcal{D}}_f v^{ef} \right.$$
$$\left. \left. +e^{-1}\epsilon_{abcde}v^c_f\hat{\mathcal{D}}^d v^{ef} - \frac{4}{3}v_{ac}v^{cd}v_{db} - \frac{1}{3}v_{ab}v^2 \right] \right\} \qquad (2.7)$$

where $c_I$ are some constant coefficients,[1] $C_{abcd}$ is the Weyl tensor which in five dimensions is

$$C^{ab}{}_{cd} = R^{ab}{}_{cd} - \frac{1}{3}\left(g_c^a R_d^b - g_d^a R_c^b - g_c^b R_d^a + g_d^b R_c^a\right) + \frac{1}{12}R\left(g_c^a g_d^b - g_d^a g_c^b\right) \qquad (2.8)$$

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[1]From the viewpoint of compactification of $D = 11$ SUGRA they are topological numbers connected to second Chern class, see [20].

and $\hat{\mathcal{D}}_a$ is the conformal covariant derivative, which when appearing linearly in (2.7) can be substituted with ordinary covariant derivative $\mathcal{D}_a$, but when taken twice produces additional curvature contributions [21]:

$$v_{ab}\hat{\mathcal{D}}^b\hat{\mathcal{D}}_c v^{ac} = v_{ab}\mathcal{D}^b\mathcal{D}_c v^{ac} + \frac{2}{3}v^{ac}v_{cb}R_a^b + \frac{1}{12}v^2 R \tag{2.9}$$

We are going to analyse extremal black hole solutions of the action obtained by combining (2.1) and (2.7):[2]

$$\mathcal{A} = \int dx^5\sqrt{-g}\mathcal{L} = \int dx^5\sqrt{-g}(\mathcal{L}_0 + \mathcal{L}_1) \tag{2.10}$$

As (2.7) is a complicated function of auxiliary fields (including derivatives) it is now impossible to integrate them out in the closed form and obtain an action which includes just the physical fields.

## 3. Near horizon geometry and entropy function formalism

The action (2.10) is quartic in derivatives and generally probably too complicated for finding complete analytical black hole solutions even in the simplest spherically symmetric case. But, if one is more modest and interested just in a near-horizon behavior (which is enough to find the entropy) of *extremal* black holes, there is a smart way to do the job - Sen's entropy function formalism [16].[3]

For five-dimensional spherically symmetric extremal black holes near-horizon geometry is expected to be $AdS_2 \times S^3$, which has $SO(2,1) \times SO(4)$ symmetry [25]. If the Lagrangian can be written in a manifestly diffeomorphism covariant and gauge invariant way (as a function of metric, Riemann tensor, covariant derivative, and gauge invariant fields, but without connections) it is expected that near the horizon the complete background should respect this symmetry. Then it follows that the only fields which can acquire non-vanishing values near the horizon are scalars $\phi_s$, (purely electric) two-forms fields $F^I$, and (purely magnetic) three-form fields $H_m$. Explicitly written:

$$ds^2 = v_1\left(-x^2 dt^2 + \frac{dx^2}{x^2}\right) + v_2\, d\Omega_3^2$$

$$\phi_s = u_s\,, \qquad\qquad\qquad s = 1,\ldots,n_s$$
$$F^I = -e^I \epsilon_A\,, \qquad\qquad\quad I = 1,\ldots,n_F$$
$$H_m = 2q_m \epsilon_S \qquad\qquad\quad m = 1,\ldots,n_H \tag{3.1}$$

where $v_{1,2}$, $u_s$, $e^I$ and $q_m$ are constants, $\epsilon_A$ and $\epsilon_S$ are induced volume-forms on $AdS_2$ and $S^3$, respectively. In case where $F^I$ ($H_m$) are gauge field strengths, $e^I$ ($q_m$) are electric field strengths (magnetic charges).

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[2]Our conventions for Newton coupling is $G_5 = \pi^2/4$ and for the string tension $\alpha' = 1$.

[3]This formalism was used recently in near-horizon analyses of a broad classes of black holes and higher dimensional objects [22]. For generalisation to rotating black holes see [23]. For comparison with SUSY entropy functions see [24].

It is important to notice that all covariant derivatives in this background are vanishing. To obtain near-horizon solutions one defines

$$f(\vec{v}, \vec{u}, \vec{e}) = \int_{S^3} \sqrt{-g}\, \mathcal{L} \tag{3.2}$$

extremization of which over $\vec{v}$ and $\vec{u}$ gives equations of motion (EOM's)

$$\frac{\partial f}{\partial v_i} = 0 \,, \qquad \frac{\partial f}{\partial u_s} = 0 \,, \tag{3.3}$$

and derivatives over $\vec{e}$ are giving (properly normalized) electric charges:

$$q_I = \frac{\partial f}{\partial e^I} \tag{3.4}$$

Finally, the entropy (equal to the Wald formula [26]) is given with

$$S_{\mathrm{BH}} = 2\pi \left( q_I\, e^I - f \right) \tag{3.5}$$

Equivalently, one can define entropy function $F$ as a Legendre transform of the function $f$ with respect to the electric fields and charges

$$F(\vec{v}, \vec{u}, \vec{e}, \vec{q}) = 2\pi \left( q_I\, e^I - f(\vec{v}, \vec{u}, \vec{e}) \right) \tag{3.6}$$

Now equations of motion are obtained by extremizing entropy function

$$0 = \frac{\partial F}{\partial v_i} \,, \qquad 0 = \frac{\partial F}{\partial u_s} \,, \qquad 0 = \frac{\partial F}{\partial e^I} \tag{3.7}$$

and the value at the extremum gives the black hole entropy

$$S_{\mathrm{BH}} = F(\vec{v}, \vec{u}, \vec{e}, \vec{q}) \qquad \text{when } \vec{v}, \vec{u}, \vec{e} \text{ satisfy (3.7)} \tag{3.8}$$

We want next to apply entropy function formalism to the $N = 2$ SUGRA from section 2. In this case for the near-horizon geometry (3.1) we explicitly have

$$ds^2 = v_1 \left( -x^2 dt^2 + \frac{dx^2}{x^2} \right) + v_2\, d\Omega_3^2$$
$$F_{\mathrm{tr}}^I(x) = -e^I \,, \qquad v_{\mathrm{tr}}(x) = V$$
$$M^I(x) = M^I \,, \qquad D(x) = D \tag{3.9}$$

where $v_i$, $e^I$, $M^I$, $V$, and $D$ are constants.

Putting (3.9) into (2.1) and (2.7) one gets

$$f_0 = \frac{1}{4}\sqrt{v_2} \left[ (\mathcal{N} + 3)(3v_1 - v_2) - 4V^2 (3\mathcal{N} + 1)\frac{v_2}{v_1} \right.$$
$$\left. + 8V\mathcal{N}_i e^i \frac{v_2}{v_1} - \mathcal{N}_{ij} e^i e^j \frac{v_2}{v_1} + D(\mathcal{N} - 1)v_1 v_2 \right] \tag{3.10}$$

and

$$f_1 = v_1 v_2^{3/2} \left\{ \frac{c_I e^I}{48} \left[ -\frac{4V^3}{3v_1^4} + \frac{DV}{3v_1^2} + \frac{V}{v_1^2} \left( \frac{1}{v_1} - \frac{1}{v_2} \right) \right] \right.$$
$$\left. + \frac{c_I M^I}{48} \left[ \frac{D^2}{12} + \frac{4V^4}{v_1^4} + \frac{1}{4} \left( \frac{1}{v_1} - \frac{1}{v_2} \right)^2 - \frac{2V^2}{3v_1^2} \left( \frac{5}{v_1} + \frac{3}{v_2} \right) \right] \right\} , \qquad (3.11)$$

correspondingly. Notice that for the background (3.9) all terms containing $\varepsilon_{abcde}$ tensor vanish. Complete function $f$ is a sum

$$f = f_0 + f_1 \qquad (3.12)$$

and EOM's near the horizon are equivalent to

$$0 = \frac{\partial f}{\partial v_1} , \qquad 0 = \frac{\partial f}{\partial v_2} , \qquad 0 = \frac{\partial f}{\partial M^I} , \qquad 0 = \frac{\partial f}{\partial V} , \qquad 0 = \frac{\partial f}{\partial D} . \qquad (3.13)$$

Notice that both $f_0$ and $f_1$ (and so $f$) are invariant on the transformation $e^I \to -e^I$, $V \to -V$, with other variables remaining the same. This symmetry follows from CPT invariance. We shall use it to obtain new solutions with $q_I \to -q_I$.

## 4. Solutions with maximal supersymmetry

We want to find near horizon solutions using entropy function formalism described in section 3. The procedure is to fix the set of electric charges $q_I$ and then solve the system of equations (3.13), (3.4) with the function $f$ given by (3.10)–(3.12). It is immediately obvious that though the system is algebraic, it is in generic case too complicated to be solved in direct manner, and that one should try to find some additional information.

Such additional information can be obtained from supersymmetry. It is known that there should be 1/2 BPS black hole solutions, for which it was shown in [27] that near the horizon supersymmetry is enhanced fully. This means that in this case we can put all variations in (2.6) to zero, which for $AdS_2 \times S^3$ background become

$$0 = \mathcal{D}_\mu \varepsilon^i + \frac{1}{2} v^{ab} \gamma_{\mu ab} \varepsilon^i - \gamma_\mu \eta^i$$
$$0 = D\varepsilon^i + 4\gamma \cdot v \, \eta^i$$
$$0 = -\frac{1}{4}\gamma \cdot F^I \varepsilon^i - M^I \eta^i$$
$$0 = \left( 3\eta^j - \gamma \cdot v \, \varepsilon^j \right) \mathcal{A}_j^\alpha \qquad (4.1)$$

Last equation fixes the spinor parameter $\eta$ to be

$$\eta^j = \frac{1}{3}(\gamma \cdot v)\varepsilon^j \qquad (4.2)$$

Using this, and the condition that $\varepsilon^i$ is (geometrical) Killing spinor, in the remaining equations one gets[4] the following conditions

$$v_2 = 4v_1 , \qquad M^I = \frac{e^I}{\sqrt{v_1}} , \qquad D = -\frac{3}{v_1} , \qquad V = \frac{3}{4}\sqrt{v_1} \qquad (4.3)$$

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[4]As the detailed derivation was already presented in [17] (solutions in the whole space) and in [18] (near horizon solutions), we shall just state the results here.

We see that conditions for full supersymmetry are so constraining that they fix everything except one unknown, which we took above to be $v_1$. To fix it, we just need one equation from (3.13). In our case the simplest is to take equation for $D$, which gives

$$v_1^{3/2} = (e)^3 - \frac{c_I e^I}{48} \tag{4.4}$$

where we used a notation

$$(e)^3 \equiv \frac{1}{6} c_{IJK} e^I e^J e^K \tag{4.5}$$

We note that higher derivative corrections violate real special geometry condition, i.e., we have now $\mathcal{N} \neq 1$.[5]

Using (4.3) and (4.4) in the expression for the entropy (3.5) one obtains

$$S_{\text{BH}} = 16\pi(e)^3 \tag{4.6}$$

Typically one is interested in expressing the results in terms of charges, not field strengths, and this is achieved by using (3.4). As shown in [17], the results can be put in compact form in the following way. We first define scaled moduli

$$\bar{M}^I \equiv \sqrt{v_1} M^I \,. \tag{4.7}$$

Solution for them is implicitly given with

$$8\, c_{IJK} \bar{M}^J \bar{M}^K = q_I + \frac{c_I}{8} \tag{4.8}$$

and the entropy (4.6) becomes

$$S_{\text{BH}} = \frac{8\pi}{3} c_{IJK} \bar{M}^I \bar{M}^J \bar{M}^K \tag{4.9}$$

A virtue of this presentation is that if one is interested only in entropy, then it is enough to consider just (4.8) and (4.9). It was shown in [28] that (4.9) agrees with the OSV conjecture [1, 29], after proper treatment of uplift from $D = 4$ to $D = 5$ is made.

We shall be especially interested in the case when prepotential is of the form

$$\mathcal{N} = \frac{1}{2} M^1 c_{ij} M^i M^j \,, \qquad i, j > 1 \tag{4.10}$$

where $c_{ij}$ is a regular matrix with an inverse $c^{ij}$. In this case, which corresponds to $K3 \times T^2$ 11-dimensional compactifications, it is easy to show that the entropy of BPS black holes is given with

$$S_{\text{BH}} = 2\pi \sqrt{\frac{1}{2} |\hat{q}_1| c^{ij} \hat{q}_i \hat{q}_j} \,, \qquad \hat{q}_I = q_I + \frac{c_I}{8} \tag{4.11}$$

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[5]We emphasize that one should be cautious in geometric interpretation of this result. Higher order corrections generally change relations between fields in the effective action and geometric moduli, and one needs field redefinitions to restore the relations. Then correctly defined moduli may still satisfy condition for real special geometry.

# 5. $\mathcal{N} = M^1 M^2 M^3$ model — heterotic string on $T^4 \times S^1$

## 5.1 BH solutions without corrections

To analyse non-BPS solutions we take a simple model with $I = 1, 2, 3$ and prepotential

$$\mathcal{N} = M^1 M^2 M^3 \; , \tag{5.1}$$

which is obtained when one compactifies 11-dimensional SUGRA on six-dimensional torus $T^6$. It is known [30, 20] that with this choice one obtains tree level effective action of heterotic string compactified on $T^4 \times S^1$ which is wounded around $S^1$.

The simplest way to see this is to do the following steps. Start with the Lagrangian in the on-shell form (2.4), use (5.1) with the condition $\mathcal{N} = 1$, introduce two independent moduli $S$ and $T$ such that

$$M^1 = S^{2/3} \; , \qquad M^2 = S^{-1/3} T^{-1} \; , \qquad M^3 = S^{-1/3} T \tag{5.2}$$

Finally, make Poincare duality transformation on the two-form gauge field $F^1$: introduce additional 2-form $B$ with the corresponding strength $H = dB$ and add to the action a term

$$\mathcal{A}_B = \frac{1}{4\pi^2} \int F^1 \wedge H = -\frac{1}{8\pi^2} \int dx^5 \sqrt{-g} F^1_{ab} (*H)^{ab} \tag{5.3}$$

where $*$ is Hodge star. If one first solves for the $B$ field, the above term just forces two-form $F^1$ to satisfy Bianchi identity, so the new action is classically equivalent to the starting one. But if one solves for the $F^1$ and puts the solution back into the action, after the dust settles one obtains that Lagrangian density takes the form

$$4\pi^2 \mathcal{L}_0 = R - \frac{1}{3} \frac{(\partial S)^2}{S^2} - \frac{(\partial T)^2}{T^2} - \frac{S^{4/3}}{12} \left( H'_{abc} \right)^2 - \frac{1}{4} S^{2/3} T^2 \left( F^2_{ab} \right)^2 - \frac{S^{2/3}}{4 T^2} \left( F^3_{ab} \right)^2 \tag{5.4}$$

where 3-form field $H'$ is defined with

$$H'_{abc} = \partial_a B_{bc} - \frac{1}{2} \left( A^2_a F^3_{bc} + A^3_a F^2_{bc} \right) + \text{(cyclic permutations of } a, b, c) \tag{5.5}$$

To get the action in an even more familiar form one performs a Weyl rescaling of the metric

$$g_{ab} \to S^{2/3} g_{ab} \tag{5.6}$$

where in the new metric Lagrangian (5.4) takes the form

$$4\pi^2 \mathcal{L}_0 = S \left[ R + \frac{(\partial S)^2}{S^2} - \frac{(\partial T)^2}{T^2} - \frac{1}{12} \left( H'_{abc} \right)^2 - \frac{T^2}{4} \left( F^2_{ab} \right)^2 - \frac{1}{4 T^2} \left( F^3_{ab} \right)^2 \right] \tag{5.7}$$

One can now check[6] that (5.4) and (5.7) are indeed lowest order (in $\alpha'$ and $g_s$) effective Lagrangians in Einstein and string frame, respectively, of the heterotic string compactified

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[6]For example by comparing with eqs. (2.2), (2.8) and (2.3) in [12]. Observe that, beside simple change in indices $1 \to 2$ and $2 \to 3$, one needs to divide gauge fields by a factor of two to get results in Sen's conventions. There is also a difference in a convention for $\alpha'$, which makes normalization of charges different.

on $T^4 \times S^1$ with the only "charges" coming from winding and momentum on $S^1$. Field $T$ plays the role of a radius of $S^1$, and field $S$ is a function of a dilaton field such that $S \sim 1/g_s^2$. This interpretation immediately forces all $M^I$ to be positive.

We are interested in finding 3-charge near-horizon solutions for BH's when the prepotential is (5.1). Applying entropy function formalism on (3.10) one easily gets:

$$v_1 = \frac{1}{4} |q_1 q_2 q_3|^{1/3} \tag{5.8}$$

$$e^I = \frac{4 v_1^{3/2}}{q_I} = \frac{1}{2 q_I} |q_1 q_2 q_3|^{1/2} \tag{5.9}$$

$$M^I = \frac{|e^I|}{\sqrt{v_1}} = \left| \frac{q_1 q_2 q_3}{q_I^2} \right|^{1/3} \tag{5.10}$$

$$v_2 = 4 v_1 \tag{5.11}$$

$$D = -\frac{1}{v_1} |\mathrm{sign}(q_1) + \mathrm{sign}(q_2) + \mathrm{sign}(q_3)| \tag{5.12}$$

$$V = \frac{\sqrt{v_1}}{4} (\mathrm{sign}(q_1) + \mathrm{sign}(q_2) + \mathrm{sign}(q_3)) \tag{5.13}$$

and the entropy is given with

$$S = 2\pi |q_1 q_2 q_3|^{1/2} \tag{5.14}$$

In fact in this case full solutions (not only near-horizon but in the whole space) were explicitly constructed in [31].

If any of charges $q_I$ vanishes, one gets singular solutions with vanishing horizon area. Such solutions correspond to small black holes. One expects that higher order (string) corrections "blow up" the horizon and make solutions regular.

## 5.2 Inclusion of SUSY corrections

We would now like to find near horizon solutions for extremal black holes when the action is extended with the supersymmetric higher derivative correction (2.7). We already saw in section 4 how this can be done for the special case of 1/2 BPS solutions, i.e., in case of non-negative charges $q_I \geq 0$. The question is could the same be done for general sets of charges.

Again, even for the simple prepotential (5.1) any attempt of direct solving of EOM's is futile. In the BPS case we used vanishing of all supersymmetry variations which gave the conditions (5.10)–(5.14), which are not affected by higher derivative correction, and that enabled us to find a complete solution. Now, for non-BPS case, we cannot use the same argument, and naive guess that (5.10), (5.14) is preserved after inclusion of correction is inconsistent with EOM's.

Intriguingly, there is something which is shared by (BPS and non-BPS) solutions (5.8)–(5.13) — the following two relations:

$$0 = D v_1 + 3 - 9 \frac{v_1}{v_2} + 4 \frac{V^2}{v_1} \tag{5.15}$$

$$0 = \frac{(D v_1)^2}{12} + 4 \left( \frac{V}{\sqrt{v_1}} \right)^4 + \frac{1}{4} \left( 1 - \frac{v_1}{v_2} \right)^2 - \frac{2}{3} \left( \frac{V}{\sqrt{v_1}} \right)^2 \left( 5 + 3 \frac{v_1}{v_2} \right) \tag{5.16}$$

The above conditions are connected with supersymmetry. The first one, when plugged in the $\mathcal{L}_0$ (2.1), makes the first bracket (multiplying $\mathcal{A}^2$) to vanish. The second condition, when plugged in the $\mathcal{L}_1$ (2.7), makes the term multiplying $c_I M^I$ to vanish. We shall return to this point in section 6.

What is important is that for (5.15) and (5.16) we needed just eqs. (5.11)–(5.13) (and, in particular, *not* eq. (5.10)). Our idea is to take (5.11)–(5.13) as an *ansatz*, plug this into all EOM's and find out is it working also in the non-BPS case. Using the CPT symmetry it is obvious that there are just two independent cases. We can choose

$$v_2 = 4v_1 \;, \qquad D = -\frac{3}{v_1} \;, \qquad V = \frac{3}{4}\sqrt{v_1} \;, \tag{5.17}$$

which corresponds to BPS case (see (4.3)), and

$$v_2 = 4v_1 \;, \qquad D = -\frac{1}{v_1} \;, \qquad V = \frac{1}{4}\sqrt{v_1} \tag{5.18}$$

Though in the lowest order (5.17) appears when all charges are positive, and (5.18) when just one charge is negative (see (5.11)–(5.13)), we shall *not* suppose a priori any condition on the charges.

For the start, let us restrict coefficients $c_I$ such that

$$c_1 \equiv 24\zeta > 0 \;, \qquad c_2 = c_3 = 0 \;. \tag{5.19}$$

This choice appears when one considers heterotic string effective action on the tree level in string coupling $g_s$, but taking into account (part of) corrections in $\alpha'$.[7] In this case we have $\zeta = 1$. For completeness, we present results for general coefficients $c_I$ in appendix A.

Let us now start with the ansatz (5.18). The EOM's can now be written in the following form:

$$b^2 b^3 e^2 e^3 = 0$$
$$b^1 b^3 e^1 e^3 = 0$$
$$b^1 b^2 e^1 e^2 = 0$$
$$4\left(b^2 b^3 - 1\right) e^2 e^3 = q_1 - \frac{\zeta}{3}$$
$$4\left(b^1 b^3 - 1\right) e^1 e^3 = q_2$$
$$4\left(b^1 b^2 - 1\right) e^1 e^2 = q_3$$
$$42v_1^{3/2} + \left(\zeta\left(6b^1 - 1\right) + 6\left(4b^1 b^2 b^3 - 3(b^1 + 1)(b^2 + 1)(b^3 + 1) + 4\right) e^2 e^3\right) e^1 = 0$$
$$18v_1^{3/2} + \left(6\left(4b^1 b^2 b^3 + (b^1 + 1)(b^2 + 1)(b^3 + 1) + 4\right) e^2 e^3 - \zeta\left(2b^1 + 5\right)\right) e^1 = 0$$
$$6v_1^{3/2} + \left(\zeta(2b^1 + 1) - 6\left(b^1 + 1\right)\left(b^2 + 1\right)\left(b^3 + 1\right) e^2 e^3\right) e^1 = 0$$
$$6v_1^{3/2} + \left(\zeta\left(10b^1 + 9\right) + 6\left(3b^1 b^2 b^3 - b^1 b^2 - b^2 b^3 - b^1 b^3 - 5(b^1 + b^2 + b^3) - 9\right) e^2 e^3\right) e^1 = 0$$

where $b^I$ are defined with

$$\bar{M}^I \equiv (1 + b^I)e^I \tag{5.20}$$

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[7]To consider corrections in $g_s$ it would be necessary also to make corrections in the prepotential (i.e., to $c_{IJK}$).

Now there are more equations than unknowns, so the system is naively overdetermined. However, not all equations are independent and the system is solvable. First notice that first three equations imply that two of $b^I$'s should vanish, which enormously simplifies solving.

Let us summarize our results. We have found that there are six branches of solutions satisfying[8] $M^I > 0$, depending on the value of the charges $q_I$.

- $q_1 > \zeta/3$, $q_2 > 0$, $q_3 < 0$

  Solutions are given with:

$$v_1 = \frac{1}{4} \left| \frac{q_2 q_3 (q_1 + \zeta/3)^2}{q_1 - \zeta/3} \right|^{1/3} \tag{5.21}$$

$$\frac{e^1}{\sqrt{v_1^3}} \left( q_1 - \frac{\zeta}{3} \right) = \frac{e^2 q_2}{\sqrt{v_1^3}} = \frac{e^3 q_3}{\sqrt{v_1^3}} = 4 \frac{q_1 - \zeta/3}{q_1 + \zeta/3} \tag{5.22}$$

$$\frac{M^3 \sqrt{v_1}}{e^3} = -\frac{q_1 + \zeta}{q_1 - \zeta/3}, \qquad \frac{M^1 \sqrt{v_1}}{e^1} = \frac{M^2 \sqrt{v_1}}{e^2} = 1 \tag{5.23}$$

  together with (5.18). The entropy is given with

$$S_{\mathrm{BH}} = 2\pi \left| q_2 q_3 \left( q_1 - \frac{\zeta}{3} \right) \right|^{1/2} \tag{5.24}$$

  For heterotic string one has $\zeta = 1$ and $q_I$ are integer, so the condition can be written also as $q_1 > 0$.

- $q_1 > \zeta/3$, $q_2 < 0$, $q_3 > 0$

  As the theory is symmetric on the exchange $(2) \leftrightarrow (3)$, the only difference from the previous case is that now we have

$$\frac{M^2 \sqrt{v_1}}{e^2} = -\frac{q_1 + \zeta}{q_1 - \zeta/3}, \qquad \frac{M^1 \sqrt{v_1}}{e^1} = \frac{M^3 \sqrt{v_1}}{e^3} = 1 \tag{5.25}$$

  and everything else is the same.

- $q_1 < -\zeta$, $q_2 > 0$, $q_3 > 0$

  Here the only difference from solutions in previous two cases is:

$$\frac{M^1 \sqrt{v_1}}{e^1} = -\frac{q_1 - \zeta/3}{q_1 + \zeta}, \qquad \frac{M^2 \sqrt{v_1}}{e^2} = \frac{M^3 \sqrt{v_1}}{e^3} = 1 \tag{5.26}$$

  For heterotic string $\zeta = 1$ the bound for $q_1$ is $q_1 < -1$.

Beside these three "normal" branches, there are additional three "strange" branches which appear for $|q_1| < \zeta/3$:

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[8]We note that, as was shown in $D = 4$ [11], that corrections can change relations between fields in the action and moduli of the compactification manifold, so one should be careful when demanding physicality conditions.

- $|q_1| < \zeta/3$, $q_2 < 0$, $q_3 < 0$

  For every of the three branches discussed above, there is an additional, mathematically connected, branch, for which the difference is that now in all branches we have $|q_1| < \zeta/3$, $q_2 < 0$, $q_3 < 0$. All formulas are the same, except that the entropy is negative

  $$S_{\text{BH}} = -2\pi \left| q_2 q_3 \left( q_1 - \frac{\zeta}{3} \right) \right|^{1/2} \tag{5.27}$$

  Additional reason why we call these solutions "strange" is the fact that electric fields and charges have opposite sign. It is questionable that there are asymptotically flat BH solutions with such near-horizon behaviour, and for the rest of the paper we shall ignore them.

Now we take the "BPS" ansatz (5.17). There is only one branch of solutions, valid for $q_{2,3} > 0$, $q_1 > -\zeta$:

- $q_1 > -\zeta$, $q_2 > 0$, $q_3 > 0$

  Solution now takes the form

  $$v_1 = \frac{1}{4} \left| \frac{q_2 q_3 (q_1 + \zeta)^2}{q_1 + 3\zeta} \right|^{1/3} \tag{5.28}$$

  $$\frac{e^1}{\sqrt{v_1^3}} (q_1 + 3\zeta) = \frac{e^2 q_2}{\sqrt{v_1^3}} = \frac{e^3 q_3}{\sqrt{v_1^3}} = 4 \frac{q_1 + 3\zeta}{q_1 + \zeta} \tag{5.29}$$

  $$\frac{M^1 \sqrt{v_1}}{e^1} = \frac{M^2 \sqrt{v_1}}{e^2} = \frac{M^3 \sqrt{v_1}}{e^3} = 1 \tag{5.30}$$

  together with (5.17). The entropy is given with

  $$S_{\text{BH}} = 2\pi \left| q_2 q_3 (q_1 + 3\zeta) \right|^{1/2} \tag{5.31}$$

  One can check that this is equal to the BPS solution from section 4 with the prepotential and $c_I$ given by (5.1) and (5.19).

Solutions for the cases when two or all three charges are negative are simply obtained by applying the CPT transformations $e^I \to -e^I$, $q^I \to -q^I$, $V \to -V$ on the solutions above.

## 5.3 Some remarks on the solutions

Let us summarize the results of section 5.2. For the prepotential (5.1) and (5.19) we have found nonsingular extremal near-horizon solutions with $AdS_2 \times S^3$ geometry for all values of charges $(q_1, q_2, q_3)$ except for some special cases. For black hole entropy we have obtained that supersymmetric higher order ($R^2$) correction just introduces a shift $q_1 \to \hat{q}_1 = q_1 + a$,

$$S_{\text{BH}} = 2\pi \sqrt{|\hat{q}_1 q_2 q_3|}$$

where $a = \pm 3, \pm 1/3$.

For the action connected with compactified heterotic string, i.e., when $\zeta = 1$ and charges are integer valued, exceptions are:

(i) $q_2 q_3 = 0$

(ii) $q_1 = 0 \ , \ \ q_2 q_3 < 0$

(iii) $q_1 = -1 \ , \ \ q_2, q_3 > 0$ (and also with reversed signs)

It is easy to show that in order to have small effective string coupling near the horizon we need $q_2 q_3 \gg 1$ which precludes case (i) (string loop corrections make $c_{2,3} \neq 0$ which regulate case (i), see Append. A). For the cases (ii) and (iii) one possibility is that regular solutions exist, but they are not given by our ansätze. But, our efforts to find numerical solutions also failed, so it is also possible that such solutions do not exist. This would not be that strange for cases (i) and (ii), as they correspond to black hole solutions which were already singular (small) with vanishing entropy before inclusion of supersymmetric $R^2$ corrections. But for the case (iii) it would be somewhat bizarre, because it would mean that higher order corrections turn nonsingular solution into singular.

Let us make a comment on a consequence of the violation of the real special geometry condition by supersymmetric higher-derivative corrections. We have seen that the example analysed in this section can be viewed as the tree-level effective action of heterotic string compactified on $T^4 \times S^1$ supplied with part of $\alpha'$ corrections. In section 5.1 we saw that in the lowest order a radius $T$ of $S^1$ was identified with $T^2 = M^3/M^2$. From (5.28)–(5.30) follows that in the BPS solution we have

$$T^2 = \frac{q_2}{q_3} \tag{5.32}$$

which is expected from T-duality $q_2 \leftrightarrow q_3$, $T \to T^{-1}$.

But, in the lowest order we also have $T^2 = M^1 (M^3)^2$, which gives

$$T^2 = \frac{q_2}{q_3} \frac{q_1 + 3}{q_1 + 1} \tag{5.33}$$

which does not satisfy T-duality. It means that relation $T^2 = M^1 (M^3)^2$ receives higher-derivative corrections.[9] That at least one of relations for $T$ is violated by corrections was of course expected from $\mathcal{N} \neq 1$.[10]

# 6. Generalisation to other prepotentials

A natural question would be to ask in what extend one can generalize construction from the previous section. In mathematical terms, the question is of validity of ansatz (5.18)

$$v_2 = 4 v_1 \ , \qquad D = -\frac{1}{v_1} \ , \qquad V = \frac{1}{4} \sqrt{v_1} \tag{6.1}$$

which we call ansatz 1, and (5.17)

$$v_2 = 4 v_1 \ , \qquad D = -\frac{3}{v_1} \ , \qquad V = \frac{3}{4} \sqrt{v_1} \ , \tag{6.2}$$

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[9]Similar observation in $D = 4$ dimensions was given in [11].

[10]Notice that for some non-BPS solutions both relations are violated.

which we call ansatz 3 (ansatz 2 and 4 are obtained by applying CPT transformation, i.e, $V \to -V$).

We have seen in section 4 that for BPS states supersymmetry directly dictates validity of ansatz 3 (and by symmetry also 4). The remaining question is how general is ansatz 1.

Putting (6.1) in EOM's one gets

$$c_{IJK}e^J e^K + 2\bar{\mathcal{N}}_I = 2\bar{\mathcal{N}}_{IJ}e^J$$

$$6\left(c_I \bar{M}^I + 168 v_1^{3/2} + 24\bar{\mathcal{N}} + 48\bar{\mathcal{N}}_{IJ}e^I e^J\right) = 7c_I e^I + 576\bar{\mathcal{N}}_I e^I$$

$$144\left(3v_1^{3/2} + 5\bar{\mathcal{N}} + 2\bar{\mathcal{N}}_{IJ}e^I e^J\right) = 3c_I e^I + 2c_I \bar{M}^I + 576\bar{\mathcal{N}}_I e^I$$

$$c_I e^I + 144\bar{\mathcal{N}} = 2\left(c_I \bar{M}^I + 72 v_1^{3/2}\right)$$

$$c_I e^I + 576\bar{\mathcal{N}}_I e^I = 10 c_I \bar{M}^I + 144 v_1^{3/2} + 432\bar{\mathcal{N}}$$

$$q_I - \frac{c_I}{72} = 4\bar{\mathcal{N}}_I - 4\bar{\mathcal{N}}_{IJ}e^J \tag{6.3}$$

and for the black hole entropy

$$S_{\text{BH}} = 4\pi\left(2\bar{\mathcal{N}} - \bar{\mathcal{N}}_{IJ}e^I e^J\right) = \frac{4\pi}{3}\hat{q}_I e^I \tag{6.4}$$

It can be shown that two equations in (6.3) are not independent. In fact, by further manipulation the system can be put in the simpler form

$$0 = c_{IJK}\left(\bar{M}^J - e^J\right)\left(\bar{M}^K - e^K\right) \tag{6.5}$$

$$\frac{c_I \bar{M}^I}{12} = c_{IJK}\left(\bar{M}^I + e^I\right)\bar{M}^J e^K \tag{6.6}$$

$$v_1^{3/2} = \frac{c_I e^I}{144} - (e)^3 \tag{6.7}$$

$$q_I - \frac{c_I}{72} = -2\,c_{IJK}e^J e^K \tag{6.8}$$

Still the above system is generically overdetermined as there is one equation more than the number of unknowns. More precisely, eqs. (6.5) and (6.6) should be compatible, and this is not happening for generic choice of parameters. One can check this, e.g., by numerically solving simultaneously (6.5) and (6.6) for random choices of $c_{IJK}$, $c_I$ and $e^I$. This means that for generic prepotentials the ansatz 1 (6.1) is not working.

However, there are cases in which the system is regular and there are physical solutions. This happens, e.g., for prepotentials of the type

$$\mathcal{N} = \frac{1}{2}M^1 c_{ij}M^i M^j \,, \qquad i,j > 1 \tag{6.9}$$

where $c_{ij}$ is a regular matrix. In this case (6.5) gives conditions

$$0 = \left(\bar{M}^1 - e^1\right)\left(\bar{M}^i - e^i\right) \,, \qquad 0 = \left(\bar{M}^i - e^i\right)c_{ij}\left(\bar{M}^j - e^j\right) \tag{6.10}$$

which has one obvious solution when $\bar{M}^i = e^i$ for all $i$. Now $\bar{M}^1$ is left undetermined, and one uses "the extra equation" (6.6) to get it. Black hole entropy becomes

$$S_{\text{BH}} = 2\pi\sqrt{\frac{1}{2}|\hat{q}_1|c^{ij}\hat{q}_i\hat{q}_j} \,, \qquad \hat{q}_I = q_I - \frac{c_I}{72} \tag{6.11}$$

where $c^{ij}$ is matrix inverse of $c_{ij}$. Again, the influence of higher order supersymmetric correction is just to shift electric charges $q_I \to \hat{q}_I$, but with the different value for the shift constant than for BPS black holes.

We have noted in section 5.2 that ansatz 1 (5.18), which gives nonsupersymmetric solutions, has some interesting relations with supersymmetry. Another way to see this is to analyse supersymmetry variations (2.6). Let us take that spinor parameters $\eta$ and $\varepsilon$ are now connected with

$$\eta^i = (\gamma \cdot v)\varepsilon^i \tag{6.12}$$

The variations (2.6) now become

$$\delta\psi_\mu^i = \left( \mathcal{D}_\mu + \frac{1}{2}v^{ab}\gamma_{\mu ab} - \gamma_\mu(\gamma \cdot v) \right)\varepsilon^i \tag{6.13}$$

$$\delta\xi^i = \left( D + 4(\gamma \cdot v)^2 \right)\varepsilon^i \tag{6.14}$$

$$\delta\Omega^{Ii} = -\left( \frac{1}{4}\gamma \cdot F^I + M^I\gamma \cdot v \right)\varepsilon^i \tag{6.15}$$

$$\delta\zeta^\alpha = 2(\gamma \cdot v)\varepsilon^j \mathcal{A}_j^\alpha \tag{6.16}$$

One can take a gauge in which $\mathcal{A}_j^\alpha = \delta_j^\alpha$, which means that last (hypermultiplet) variation (6.16) is now nonvanishing. But, it is easy to see that for ansatz 1 (and when $\epsilon^i$ is Killing spinor) variations (6.13) and (6.14) are vanishing. Also, we have seen that solutions we have been explicitly constructed have the property that for all values of the index $I$ *except one* (which we denote $J$) we had

$$\bar{M}^I = e^I \qquad I \neq J \tag{6.17}$$

From this follows that all variations (6.15) except the one for $I = J$ are also vanishing. One possible explanation for such partial vanishing of variations could be that our non-BPS states of $N = 2$ SUGRA are connected with BPS states of some theory with higher (e.g., $N = 4$) supersymmetry.

## 7. Gauss-Bonnet correction

It is known that in some cases of black holes in $D = 4$ Gauss-Bonnet term somehow effectively takes into account all $\alpha'$ string corrections. Let us now investigate what is happening in $D = 5$. This means that we now add as $R^2$ correction to the 0$^{\text{th}}$ order Lagrangian (2.1) instead of (2.7) just the term proportional to the Gauss-Bonnet density:

$$\mathcal{L}_{\text{GB}} = \frac{1}{4\pi^2}\frac{1}{8}\frac{c_I M^I}{24}\left( R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2 \right) \tag{7.1}$$

To apply entropy function formalism we start with

$$f = f_0 + f_{\text{GB}} \tag{7.2}$$

where $f_0$ is again given in (3.10) and $f_{\text{GB}}$ is

$$f_{\text{GB}} = -\frac{3}{2}\sqrt{v_2}\frac{c_I M^I}{24} \tag{7.3}$$

Strictly speaking, we have taken just (part of) first order correction in $\alpha'$, so normally we would expect the above action to give us at best just the first order correction in entropy. This we obtain by putting $0^{\text{th}}$-order solution in the expression

$$\Delta S_{\text{BH}} = -2\pi \Delta f \tag{7.4}$$

where $\Delta f$ is $1^{\text{st}}$-order correction in $f$. It is easy to show that for the BPS $0^{\text{th}}$-order solution (4.3) one obtains the same result for supersymmetric (3.11) and Gauss-Bonnet (7.3) corrections, which can be written in a form:

$$\Delta S_{\text{BH}} = 6\pi \frac{c_I e^I}{24} \tag{7.5}$$

It was noted in [17] that for compactifications on elliptically fibred Calabi-Yau (7.5) agrees with the correction of microscopic entropy proposed earlier by Vafa [32]. We note that for non-BPS black holes already first $\alpha'$ correction to entropy is different for SUSY and Gauss-Bonnet case.

From experience in $D = 4$ one could be tempted to suppose that SUSY and Gauss-Bonnet solutions are exactly (not just perturbatively) equal. However, this is not true anymore in $D = 5$. The simplest way to see this is to analyse opposite extreme where one of the charges is zero (small black holes). To explicitly show the difference let us analyse models of the type (obtained from $K3 \times T^2$ compactifications of $D = 11$ SUGRA)

$$\mathcal{N} = \frac{1}{2} M^1 c_{ij} M^i M^j , \qquad c_i = 0 , \qquad i, j > 1 \tag{7.6}$$

in the case where $q_1 = 0$. For the Gauss-Bonnet correction, application of entropy function formalism of section 3 on (7.2) gives for the entropy (see appendix B)

$$S_{\text{GB}} = 4\pi \sqrt{\frac{1}{2} \frac{c_1}{24} q_i c^{ij} q_j} \tag{7.7}$$

where $c^{ij}$ is the matrix inverse of $c_{ij}$. On the other hand, from (4.11) follows that for the supersymmetric correction in the BPS case one gets

$$S_{\text{SUSY}} = 2\pi \sqrt{\frac{3}{2} \frac{c_1}{24} q_i c^{ij} q_j} \tag{7.8}$$

which is differing from (7.7) by a factor of $2/\sqrt{3}$.

In [33] some of the models of this type were analysed from microscopic point of view and the obtained entropy of small black holes agrees with the Gauss-Bonnet result (7.7).

Now, the fact that simple Gauss-Bonnet correction is giving the correct results for BPS black hole entropy in both extremes, $q_1 = 0$ and $q_1 \gg 1$, is enough to wonder could it be that it gives the correct microscopic entropy for all $q_1 \geq 0$ (as it gives for 4 and 8-charge black holes in $D = 4$). Analytical results, with details of calculation, for the generic matrix $c_{ij}$ and charge $q_3$ are presented in appendix B.

Here we shall present results for the specific case, already mentioned in section 5, of the heterotic string compactified on $T^4 \times S^1$. Tree-level (in $g_s$) effective action is defined with

$$\mathcal{N} = M^1 M^2 M^3 , \qquad c_1 = 24 , \qquad c_2 = c_3 = 0 . \tag{7.9}$$

Matrix $c_{ij}$ is obviously here given with

$$c_{12} = c_{21} = 1 , \qquad c_{11} = c_{22} = 0 \tag{7.10}$$

As the simple Gauss-Bonnet correction (7.1) does not contain auxiliary fields, we can integrate them out in the same way as it was done in the lowest-order case in section 5.1. For independent moduli we again use

$$S \equiv (M^1)^{3/2} \qquad T \equiv \tilde{M}^2 = S^{1/3} M^2 \tag{7.11}$$

It appears that it is easier to work in string frame, where the $0^{\text{th}}$ order action is given in (5.7), and the correction (7.1) is now

$$\mathcal{L}_{\text{GB}} = \frac{1}{4\pi^2} \frac{S}{8} \left( R_{abcd} R^{abcd} - 4 R_{ab} R^{ab} + R^2 \right) + \text{ (terms containing } \partial_a S) \tag{7.12}$$

We are going to be interested in near-horizon region where all covariant derivatives, including $\partial_a S$, vanish, so we can again just keep Gauss-Bonnet density term.

Application of (3.1) here gives that solution near the horizon has the form

$$ds^2 = v_1 \left( -x^2 dt^2 + \frac{dx^2}{x^2} \right) + v_2 \, d\Omega_3^2$$
$$S(x) = S , \qquad T(x) = T$$
$$F_{\text{tr}}^{(i)}(x) = -e_i , \qquad i = 2, 3$$
$$H_{mnr} = 2 q_1 \sqrt{h_S} \, \varepsilon_{mnr} \tag{7.13}$$

where $\varepsilon_{mnr}$ is totally antisymmetric tensor with $\varepsilon_{234} = 1$. Observe that $q_1$ is now a magnetic charge. Using this in (5.7) and (7.12) gives

$$f = \frac{1}{2} v_1 v_2^{3/2} S \left( -\frac{2}{v_1} + \frac{6}{v_2} + \frac{T^2 e_2^2}{2 v_1^2} + \frac{e_3^2}{2 T^2 v_1^2} - \frac{2 q_1^2}{v_2^3} - \frac{3}{v_1 v_2} \right) \tag{7.14}$$

Following the entropy function formalism we need to solve the system of equations

$$0 = \frac{\partial f}{\partial v_1} , \qquad 0 = \frac{\partial f}{\partial v_2} , \qquad 0 = \frac{\partial f}{\partial S} , \qquad 0 = \frac{\partial f}{\partial T} , \qquad q_2 = \frac{\partial f}{\partial e_2} , \qquad q_3 = \frac{\partial f}{\partial e_3} \tag{7.15}$$

After some straightforward algebra we obtain

$$T^2 = \left| \frac{q_2}{q_3} \right| \tag{7.16}$$

which is the same as without the correction and respecting T-duality. Also

$$v_1 = \frac{v_2}{4} + \frac{1}{8} , \qquad S = \frac{1}{v_2} \sqrt{\frac{2 v_2 + 1}{2 v_2 + 3}} \sqrt{|q_2 q_3|} . \tag{7.17}$$

Here $v_2$ is the real root of a cubic equation

$$0 = x^3 - \frac{3}{2} x^2 - q_1^2 x - \frac{q_1^2}{2} \tag{7.18}$$

which, explicitly written, is

$$v_2 = \frac{1}{2} + \frac{(1 + i\sqrt{3})(4q_1^2 + 3)}{4\,3^{1/3}\left(-9 - 36q_1^2 + 2\sqrt{3}\sqrt{27q_1^2 + 72q_1^4 - 16q_1^6}\right)^{1/3}}$$
$$+ \frac{(1 - i\sqrt{3})\left(-9 - 36q_1^2 + 2\sqrt{3}\sqrt{27q_1^2 + 72q_1^4 - 16q_1^6}\right)^{1/3}}{4\,3^{2/3}} \tag{7.19}$$

For the macroscopic black hole entropy we obtain

$$S_{\mathrm{BH}} = 4\pi\sqrt{|q_2 q_3|}\sqrt{v_1 + \frac{3}{2}\frac{v_1}{v_2}} \tag{7.20}$$

It would be interesting to compare this result with the statistical entropy of BPS states (correspondingly charged) in heterotic string theory. Unfortunately, this result is still not known.

For small 2-charge black holes $q_1 = 0$, and the solution further simplifies to

$$v_1 = \frac{v_2}{3} = \frac{1}{2} \tag{7.21}$$

which gives for the entropy of small black holes

$$S_{\mathrm{BH}} = 4\pi\sqrt{|q_2 q_3|} \tag{7.22}$$

This solution was already obtained in [12] by starting at the beginning with $q_1 = 0$.[11]

## 8. Conclusion and outlook

We have shown that for some prepotentials, including important family obtained with $K3 \times T^2$ compactifications of 11-dimensional SUGRA, one can find non-BPS spherically symmetric extremal black hole near horizon solutions. In particular, for the simple example of so called $STU$ theory we have explicitly constructed solutions for all values of charges with the exception of some small black holes where one of the charges is equal to 0 or $\pm 1$.

One of the ideas was to compare results with the ones obtained by taking $R^2$ correction to be just given with Gauss-Bonnet density, and especially to analyse cases when the actions are connected with string compactifications, like e.g., heterotic string on $K3 \times S^1$, where for some instances one can find statistical entropies. Though for Gauss-Bonnet correction (which manifestly breaks SUSY) it was not possible to calculate entropy in a closed form for generic prepotentials, on some examples we have explicitly shown that in $D = 5$, contrary to $D = 4$ examples, black hole entropy is different from the one obtained using supersymmetric correction (BPS or non-BPS case). Interestingly, first order corrections to entropy of BPS black holes are the same for all prepotentials, and are in agreement with the result for statistical entropy for elliptically fibred Calabi-Yau compactification [32].

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[11]Notice that we are using $\alpha' = 1$ convention, and in [12] it is $\alpha' = 16$. One can use the results from [13] to make connection between conventions.

For the $K3 \times T^2$ compactifications of $D = 11$ SUGRA (which includes $K3 \times S^1$ compactification of heterotic string) we have found explicit formula for the black hole entropy in the case of Gauss-Bonnet correction. Unfortunately, expression for statistical entropy for generic values of charges is still not known, but there are examples for which statistical entropy of BPS states corresponding to *small* black holes is known [33]. We have obtained that Gauss-Bonnet correction leads to the macroscopic entropy equal to statistical, contrary to supersymmetric correction which leads to different result. This result favors Gauss-Bonnet correction. On the other hand, for *large* black holes, it is the supersymmetric result (4.11) which agrees with OSV conjecture properly uplifted to $D = 5$ [28]. We propose to resolve this issue perturbatively by calculating $\alpha'^2$ correction for 3-charge black holes in heterotic string theory compactified on $K3 \times S^1$ using methods of [34]. Calculation is underway and results will be presented elsewhere [35].

It is known that theories in which higher curvature correction are given by (extended) Gauss-Bonnet densities have special properties, some of which are unique. Beside familiar ones (equations of motion are "normal" second order, in flat space and some other backgrounds they are free of ghosts and other spurious states, have well defined boundary terms and variational problem, first and second order formalisms are classically equivalent, extended Gauss-Bonnet densities have topological origin and are related to anomalies, etc), they also appear special in the approaches where black hole horizon is treated as a boundary and entropy is a consequence of the broken diffeomorphisms by the boundary condition [36]. It would be interesting to understand in which way this is connected with the observed fact that these terms effectively encode a lot of near-horizon properties for a class of BPS black holes in string theory.

## Acknowledgments

## A. Solutions for $\mathcal{N} = M^1 M^2 M^3$ but general $c_I$

In this appendix we consider actions with

$$\mathcal{N} = M^1 M^2 M^3 \tag{A.1}$$

but arbitrary coefficients $c_I$. Let us define

$$\zeta_I \equiv \frac{c_I}{24} \tag{A.2}$$

and for simplicity restrict to $\zeta_I > 0$. We shall concentrate on non-BPS solutions and ansatz 1. For this case we can specialize the general expression for relation between electric charges and field strengths (6.8) as follows

$$\hat{q}_1 = -4e^2 e^3 \,, \quad \hat{q}_2 = -4e^3 e^1 \,, \quad \hat{q}_3 = -4e^1 e^2 \tag{A.3}$$

where we introduced shifted charges

$$\hat{q}_I \equiv q_I - \frac{\zeta_I}{3} \tag{A.4}$$

From here follow also simple relations

$$\frac{e^1 \hat{q}_1}{4} = \frac{e^2 \hat{q}_2}{4} = \frac{e^3 \hat{q}_3}{4} = -(e)^3 \tag{A.5}$$

We introduce definition

$$A_i = \frac{M^i}{e^i}\sqrt{v_1}, \quad i = 1, 2, 3 \tag{A.6}$$

The corresponding system of equations then follows from equation (6.3)

$$(-1 + A_2)(-1 + A_3)e^2 e^3 = 0 \tag{A.7}$$

$$(-1 + A_1)(-1 + A_3)e^1 e^3 = 0 \tag{A.8}$$

$$(-1 + A_1)(-1 + A_2)e^1 e^2 = 0 \tag{A.9}$$

$$6(7v_1^{3/2} + A_1 e^1((4 - 4A_2 + (-4 + A_2)A_3)e^2 e^3 + \zeta_1) +$$
$$+ A_2 e^2(-4(-1 + A_3)e^1 e^3 + \zeta_2) + A_3 e^3(4e^1 e^2 + \zeta_3)) = 7(e^1\zeta_1 + e^2\zeta_2 + e^3\zeta_3) \tag{A.10}$$

$$18v_1^{3/2} + 2e^1(3(4A_3 + A_1(4 - 4A_3 + A_2(-4 + 5A_3)))e^2 e^3 - A_1\zeta_1) =$$
$$= 3e^1\zeta_1 + 3e^2\zeta_2 + 2A_2 e^2(12(-1 + A_3)e^1 e^3 + \zeta_2) + (3 + 2A_3)e^3\zeta_3 \tag{A.11}$$

$$6v_1^{3/2} + 2A_1 e^1(-3A_2 A_3 e^2 e^3 + \zeta_1) + (-1 + A_2)e^2\zeta_2 + (-1 + A_3)e^3\zeta_3 = e^1\zeta_1 \tag{A.12}$$

$$6v_1^{3/2} + 2e^1(3(-4A_2 A_3 + A_1(-4A_3 + A_2(-4 + 3A_3)))e^2 e^3 + 5A_1\zeta_1) +$$
$$+ 10A_2 e^2\zeta_2 + (-1 + 10A_3)e^3\zeta_3 = e^1\zeta_1 + e^2\zeta_2 \tag{A.13}$$

We shall again find solutions with one negative and two positive shifted charges, and "strange" solutions with all shifted charges negative.

- $\hat{q}_1 < -4\zeta_1/3$ , $\quad \hat{q}_2 > 0$ , $\quad \hat{q}_3 > 0$

We first describe solutions with one charge negative, e. g., $q_1$. Then

$$\sqrt{v_1} = \frac{1}{2} \frac{\left(-Q_{(3)}^2\right)^{1/6}}{\hat{q}_1^{1/6} \hat{q}_2^{1/6} \hat{q}_3^{1/6}} \tag{A.14}$$

$$Q_{(3)} \equiv \hat{q}_1 \hat{q}_2 \hat{q}_3 + \frac{2}{3}\left(\zeta_1 \hat{q}_2 \hat{q}_3 + \zeta_2 \hat{q}_1 \hat{q}_3 + \zeta_3 \hat{q}_1 \hat{q}_2\right) \tag{A.15}$$

$$A_1 = -\frac{(4\zeta_3 \hat{q}_2 + (4\zeta_2 + 3q_2)\hat{q}_3)\hat{q}_1}{(4\zeta_1 + 3\hat{q}_1)\hat{q}_2 \hat{q}_3}, \qquad A_2 = 1, \qquad A_3 = 1 \tag{A.16}$$

$$M^1 = \frac{1}{48v_1^2} \frac{Q_{(3)}}{\hat{q}_2 \hat{q}_3} \frac{4\zeta_3 \hat{q}_2 + (4\zeta_2 + 3\hat{q}_2)\hat{q}_3}{\hat{q}_1 + \frac{4}{3}\zeta_1} \tag{A.17}$$

$$M^2 = 4v_1 \frac{\hat{q}_1 \hat{q}_3}{Q_{(3)}}, \qquad M^3 = 4v_1 \frac{\hat{q}_1 \hat{q}_2}{Q_{(3)}} \tag{A.18}$$

Here we are able to impose positivity restriction,

$$M^i > 0, \qquad i = 1, 2, 3 \tag{A.19}$$

$$M^2 > 0 \Rightarrow \frac{\hat{q}_3}{Q_{(3)}} < 0 \Rightarrow Q_{(3)} < 0 \tag{A.20}$$

$M^3 > 0$ is automatically satisfied.

Consider now $M^1 > 0$. Note first that $Q_3/\hat{q}_2\hat{q}_3 < 0$. Thus we obtain that

$$\frac{4\zeta_3\hat{q}_2 + (4\zeta_2 + 3\hat{q}_2)\hat{q}_3}{\hat{q}_1 + \frac{4}{3}\zeta_1} < 0 \tag{A.21}$$

But numerator is positive so we get

$$\hat{q}_1 + \frac{4}{3}\zeta_1 < 0 \tag{A.22}$$

Note that under mentioned restrictions the property $Q_{(3)} < 0$ is indeed satisfied. In fact

$$\begin{aligned}
Q_{(3)} &= \hat{q}_1\hat{q}_2\hat{q}_3 + \frac{2}{3}(\zeta_1\hat{q}_2\hat{q}_3 + \zeta_2\hat{q}_1\hat{q}_3 + \zeta_3\hat{q}_1\hat{q}_2) \\
&< -\frac{4}{3}\zeta_1\hat{q}_2\hat{q}_3 + \frac{2}{3}(\zeta_1\hat{q}_2\hat{q}_3 + \zeta_2\hat{q}_1\hat{q}_3 + \zeta_3\hat{q}_1\hat{q}_2) \\
&= -\frac{2}{3}\zeta_1\hat{q}_2\hat{q}_3 + \frac{2}{3}(\zeta_2\hat{q}_1\hat{q}_3 + \zeta_3\hat{q}_1\hat{q}_2) < 0
\end{aligned} \tag{A.23}$$

Let us find entropy. We shall use (3.5), (3.12) and (A.6) to obtain:

$$F = 8\pi(e)^3\{A_1A_2A_3 - (A_1 + A_2 + A_3)\} \tag{A.24}$$

But $A_2 = A_3 = 1$ so

$$S_{\mathrm{BH}} = -16\pi(e)^3 \tag{A.25}$$

From the explicit form of the solution (A.14)–(A.18) we have

$$S_{\mathrm{BH}} = 2\pi \, \mathrm{sign}\left(\frac{\hat{q}_1\hat{q}_2\hat{q}_3}{Q_{(3)}}\right)\sqrt{-\hat{q}_1\hat{q}_2\hat{q}_3} \tag{A.26}$$

But $\mathrm{sign}\left(\hat{q}_1\hat{q}_2\hat{q}_3/Q_{(3)}\right) = +1$ so

$$S_{\mathrm{BH}} = 2\pi\sqrt{|\hat{q}_1\hat{q}_2\hat{q}_3|} \tag{A.27}$$

We finally conclude that presented solution is valid for

$$\hat{q}_3 > 0, \ \hat{q}_1 < -\frac{4}{3}\zeta_1, \ \hat{q}_2 > 0 \tag{A.28}$$

completely analogous to the first case in section 5.2.

• $\hat{q}_1 > 0, \ \hat{q}_2 < -\frac{4}{3}\zeta_2, \ \hat{q}_3 > 0$

In this case all relations can be obtained from previous case with exchange $(1) \leftrightarrow (2)$.

- $\hat{q}_1 > 0 \,, \ \hat{q}_2 > 0 \,, \ \hat{q}_3 < -\frac{4}{3}\zeta_3$

Analogously this case can be obtained from the first case with interchange $(1) \leftrightarrow (3)$.

In addition to described 3 "normal" branches there are also 3 "strange" branches which give negative entropy:

$$S_{\mathrm{BH}} = -2\pi\sqrt{|\hat{q}_1\hat{q}_2\hat{q}_3|} \tag{A.29}$$

Such a solution may occur only if all $\hat{q}_I$'s are negative.

## B. Gauss-Bonnet correction in $K3 \times T^2$ compactifications

In this appendix we give the proof of the eqs. (7.7) and (7.20). We consider the actions with

$$\mathcal{N} = \frac{1}{2}M^1 c_{ij}M^i M^j \,, \qquad c_1 \equiv 24\zeta \,, \qquad c_i = 0 \,, \qquad i,j > 1 \tag{B.1}$$

and when the higher order correction is proportional to the Gauss-Bonnet density, i.e., it is given with (7.1). For such corrections one can integrate out auxiliary fields in the same manner as when there is no correction, and pass to the on-shell form of the action which is now given with (2.4) and (7.1), with the condition for real special geometry $\mathcal{N} = 1$ (which is here *not* violated by higher order Gauss-Bonnet corrections) implicitly understood.

Now, before going to a hard work, it is convenient to do following transformations (which is a generalisation of what we did in section 5.1). First we introduce scaled moduli $\bar{M}^i$ and the dilaton $S$

$$S = (M^1)^{3/2} \qquad \tilde{M}^i = S^{1/3}M^i \tag{B.2}$$

for which the real special geometry condition now reads

$$\frac{1}{2}c_{ij}\tilde{M}^i\tilde{M}^j = 1 \tag{B.3}$$

This condition fixes one of $\tilde{M}^i$'s. Then we make Poincare duality transformation (5.3) which replaces two-form gauge field strength $F^1$ with its dual 3-form strength $H$. Finally, we pass to the string frame metric by Weyl rescaling

$$g_{ab} \to S^{2/3}g_{ab} \,. \tag{B.4}$$

Again, we are interested in $AdS_2 \times S^3$ backgrounds which in the present case requires

$$ds^2 = v_1\left(-x^2 dt^2 + \frac{dx^2}{x^2}\right) + v_2\, d\Omega_3^2$$
$$F_{\mathrm{tr}}^i(x) = -e^i \,, \qquad H_{mnr} = 2q_1\sqrt{h_S}\,\varepsilon_{mnr}$$
$$\tilde{M}^i(x) = \tilde{M}^i \,, \qquad S(x) = S \tag{B.5}$$

where $\varepsilon_{mnr}$ is totally antisymmetric tensor satisfying $\varepsilon_{234} = 1$. Observe that $q_1$ now plays the role of *magnetic* charge. We apply entropy function formalism of section 3. Function $f$ is now

$$f = \frac{1}{2}v_1 v_2^{3/2} S\left(-\frac{2}{v_1} + \frac{6}{v_2} + \frac{1}{v_1^2}\tilde{G}_{ij}e^i e^j - \frac{2q_1^2}{v_2^3} - \frac{3\zeta}{v_1 v_2}\right) \tag{B.6}$$

where $\tilde{G}_{ij}$ is given with

$$\tilde{G}_{ij} = \frac{1}{2}\left(c_{ik}\tilde{M}^k c_{jl}\tilde{M}^l - c_{ij}\right) \tag{B.7}$$

To obtain solutions we need to solve extremization equations

$$0 = \frac{\partial f}{\partial v_1}, \qquad 0 = \frac{\partial f}{\partial v_2}, \qquad 0 = \frac{\partial f}{\partial S}, \qquad 0 = \frac{\partial f}{\partial \tilde{M}^i}, \qquad q_i = \frac{\partial f}{\partial e^i} \tag{B.8}$$

From the third equation (for $S$) one obtains that $f$ is vanishing. This allows us to solve immediately first two equations (for $v_1$ and $v_2$) and obtain

$$v_1 = \frac{v_2}{4} + \frac{\zeta}{8} \tag{B.9}$$

where $v_2$ is the real positive root of a cubic equation

$$0 = x^3 - \frac{3}{2}\zeta x^2 - q_1^2 x - \frac{\zeta}{2}q_1^2 \tag{B.10}$$

which is, explicitly written,

$$v_2 = \frac{\zeta}{2} + \frac{(1 + i\sqrt{3})(4q_1^2 + 3\zeta^2)}{4\,3^{1/3}\left(-9\zeta^3 - 36\zeta q_1^2 + 2\sqrt{3}\sqrt{27\zeta^4 q_1^2 + 72\zeta^2 q_1^4 - 16q_1^6}\right)^{1/3}}$$

$$+ \frac{(1 - i\sqrt{3})\left(-9\zeta^3 - 36\zeta q_1^2 + 2\sqrt{3}\sqrt{27\zeta^4 q_1^2 + 72\zeta^2 q_1^4 - 16q_1^6}\right)^{1/3}}{4\,3^{2/3}} \tag{B.11}$$

Next one can solve equations for $\tilde{M}^i$. Note that one of them is not independent because of (B.3), but this can be easily treated, e.g., by using Lagrange multiplier method. One obtains

$$0 = c_{ij}e^i\tilde{M}^j\left[c_{nk}\tilde{M}^k\left(c_{ij}e^i\tilde{M}^j\right) - 2c_{nk}e^k\right] \tag{B.12}$$

from which follow conditions

$$0 = c_{ij}e^i\tilde{M}^j \qquad \text{or} \qquad (c_{nk}\tilde{M}^k)(c_{ij}e^i\tilde{M}^j) = 2c_{nk}e^k \tag{B.13}$$

From the third equation in (B.8) (for $S$) we obtain

$$\tilde{G}_{ij}e^i e^j = v_1 + \frac{3}{2}\zeta\frac{v_1}{v_2} \tag{B.14}$$

Last equation in (B.8), which defines electric charges, gives

$$q_i = S\frac{v_2^{3/2}}{v_1}\tilde{G}_{ij}e^j \tag{B.15}$$

which, together with (B.7) and (B.13) gives

$$q_i = \mp S\frac{v_2^{3/2}}{v_1}c_{ij}e^j \tag{B.16}$$

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where the upper (lower) sign is when first (second) condition in (B.13) applies. For the entropy we need $q_i e^i$, which from (B.15) is

$$q_i e^i = \frac{v_2^{3/2}}{v_1} S \tilde{G}_{ij} e^i e^j \tag{B.17}$$

We need a solution for the dilaton $S$ which is obtained by contracting (B.16) with $q_k c^{ki}$. The result is

$$S = \frac{v_1}{v_2^{3/2}} \left| \frac{2 q_i c^{ij} q_j}{\tilde{G}_{ij} e^i e^j} \right|^{1/2} \tag{B.18}$$

Using this in (B.17) we finally get for the black hole entropy

$$S_{\text{BH}} = 2\pi q_i e^i = 4\pi \sqrt{\tilde{G}_{ij} e^i e^j} \sqrt{\frac{1}{2} |q_i c^{ij} q_j|} = 4\pi \sqrt{v_1 + \frac{3}{2} \zeta \frac{v_1}{v_2}} \sqrt{\frac{1}{2} |q_i c^{ij} q_j|} \tag{B.19}$$

where $v_1$ and $v_2$ are functions of $q_1$ and $\zeta$ given in (B.9) and (B.11). Observe that here entropy is nontrivial function of charge $q_1$ (obtained by solving cubic equation), contrary to the case of SUSY corrections which just introduce a constant shifts.

For small black holes, i.e., when $q_1 = 0$, eqs. (B.9), (B.11) and (B.14) simplify to

$$v_1 = \frac{v_2}{3} = \frac{\zeta}{2}, \qquad \tilde{G}_{ij} e^i e^j = \zeta \tag{B.20}$$

Plugging this in (B.19) gives for the entropy

$$S_{\text{BH}} = 4\pi \sqrt{\frac{\zeta}{2} |q_i c^{ij} q_j|} \qquad \text{for} \ \ q_1 = 0 \tag{B.21}$$

which is exactly (7.7).

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